Sketch of Homework 8 Solutions

#1 If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a homeo., $\mathbb{R}^m - x \approx \mathbb{R}^n - f(x)$. But $\mathbb{R}^m - x \equiv S^{m-1}$ and $\mathbb{R}^n - f(x) \equiv S^{n-1}$. Also if $\mathbb{R}^{m*}$ is the one point compactification of $\mathbb{R}^m$, then $\mathbb{R}^{m*} \approx S^m$, so $\mathbb{R}^m \approx \mathbb{R}^n \Rightarrow S^m \approx S^n$.

#2 Let $h : E^n \to E^n$ be a homeo. and $x \in S^{n-1}$. Then $E^n - x \approx E^n - h(x)$. If $h(x) \notin S^{n-1}$. Then (E^n, x) and $(E^n, h(x))$ have different local homology. $\therefore h(S^{n-1}) \subseteq S^{n-1}$. Similarly $h^{-1}(S^{n-1}) \subseteq S^{n-1}$. $\therefore h|S^{n-1} : S^{n-1} \to S^{n-1}$ is a homeo.

#3 Suppose $p \in S^{n-1}$ and $p \notin f(S^{n-1})$ Then f can be factored

$$S^{n-1} \xrightarrow{f'} S^{n-1} - p \xrightarrow{\subseteq} S^{n-1}$$

But $S^{n-1} - p$ is contractible. $\therefore f \simeq$ constant so $\deg f = 0$

#4 (a) If $p \in (E^n)^o$, $E^n - p \equiv S^{n-1}$. $\therefore H_n(E^n, E^n - p) = \mathbb{Z}$. others $= 0$

(b) If $p \in S^{n-1}$, $E^n - p \equiv p$, $\therefore H_q(E^n, E^n - p) = 0$ all q.

p

#5 M m-manifold, N n-manifold Assume $f : M \to N$ homeo. and $p \in M$. Then $H_q(M, M-p) \approx H_q(N, N - f(p))$. As in #4

$$H_q(M, M-p) = \begin{cases} \mathbb{Z} & q = m \\ 0 & \text{otherwise} \end{cases} \qquad H_q(N, N - f(p)) = \begin{cases} \mathbb{Z} & q = n \\ 0 & \text{otherwise} \end{cases}$$

#6 These are both manifold with boundary. Use local homology to show a homeo $MS \to A$ maps $\dot{MS}$ homeo to $\dot{A}$. But $\dot{MS} \approx S^1$, $\dot{A} \approx S^1 \sqcup S^1$.

#7 (b) The five lemma.

#8 Let $X = Y = MS$ Let A be the central circle and let B be the boundary of Y ($B \approx S^1$). A is dr of X so $H_q(X, A) = 0$. Show $H_1(Y, B) \neq 0$ as follows. Consider the exact sequence

$$\begin{array}{ccccccc} H_1(B) & \xrightarrow{i_*} & H_1(Y) & \xrightarrow{j_*} & H_1(Y,B) & \longrightarrow & \tilde{H}_0(B) \\ \| & & \| & & & & \| \\ \mathbb{Z} & & \mathbb{Z} & & & & 0 \end{array}$$

Since $\tilde{H}_0(B)=0$, j_* is onto. Show i_* is multiplication by ± 2 (for this consider $H_1(B) \xrightarrow{i_*} H_1(Y) \xrightarrow[\approx]{r_*} H_1(A)$ where r is the dr.

Show $r_* i_*$ is multiplication by ± 2.) $\therefore H_1(Y,B) \approx \mathbb{Z}_2$.

#9 (a) Let $a: S^n \to S^n$ be the antipodal map $\deg(af) = (-1)^{n+1}\deg f \neq (-1)^{n+1}$ $\therefore$ af has a fixed point x. $f(x) = -x$.

(b) n odd $\therefore (-1)^{n+1} = 1$ $\therefore \deg f \neq (-1)^{n+1}$. $\therefore$ f has a fixed point. By (a) f sends some point to its antipode.

(c) Similar to (b)

#10 (a) n odd, $f = id$.

(b) n even, $f =$ reflection across a hyperplane $= T_1^n$

#12 Suffices to show hypothesis of the excision axiom is equivalent to hypotheses of #12.

$\Rightarrow$ Given $X = X_1^\circ \cup X_2^\circ$ Set $X_2 = A$, $U = \complement X_1 = X - X_1$

Check $\bar{U} \subseteq A^\circ$

$\Leftarrow$ Given $\bar{U} \subseteq A^\circ$ Set $X_2 = A$, $X_1 = \complement U$ then $X_1^\circ \cup X_2^\circ = X$

#13
$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "i"] \arrow[d, "\alpha"] & B \arrow[r, "p"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "i'"] & B' \arrow[r, "p'"] & C' \arrow[r] & 0 \end{tikzcd}$$

consider also

$$\begin{tikzcd} \operatorname{Ker}\beta \arrow[r, "\tilde{p}"] \arrow[d, "u"] & \operatorname{Ker}\gamma \arrow[d, "v"] \\ B \arrow[r, "p"] & C \end{tikzcd}$$

Define $\Delta : \operatorname{Ker}\gamma \to \operatorname{coker}\alpha$: $c \in \operatorname{Ker}\gamma$, $vc = pb$ some $b \in B$

$p'\beta b = 0$, $\beta b = i'a'$ some a'. Let $\Delta c = \nu a'$ where $\nu : A' \to \operatorname{coker}\alpha$ is projection. Show Δ well-defined.

Consider
$$\operatorname{Ker}\beta \xrightarrow{\tilde{p}} \operatorname{Ker}\gamma \xrightarrow{\Delta} \operatorname{coker}\alpha \xrightarrow{\hat{i}'} \operatorname{coker}\beta$$

where $\hat{i}'$ induced by i'. Will show (a) $\operatorname{Ker}\Delta \subseteq \operatorname{Im}\tilde{p}$ (b) $\operatorname{Ker}\hat{i}' \subseteq \operatorname{Im}\Delta$. (a) Suppose $\Delta c = 0$ $\therefore \nu a' = 0$ $\therefore a' = \alpha a$ some $a \in A$

$\beta i a = \iota' \alpha a = \iota' a' = \beta b \quad \therefore b - ia \in \operatorname{Ker}\beta$

$\nu \tilde{\rho}(b - ia) = \rho u(b - ia) = \rho(b - ia) = \rho b = \nu c$

$c = \tilde{\rho}(b - ia), \quad b - ia \in \operatorname{Ker}\beta.$

(b) $\hat{\iota}' \nu a' = 0 \quad \therefore \iota' a' \in \beta B \qquad \iota' a' = \beta b$ some $b \in B$ let

$c = \rho b \qquad \gamma c = \gamma \rho b = \rho' \beta b = \rho' \iota' a' = 0 \qquad c \in \operatorname{Ker}\gamma \quad \Delta c = \nu a'$